Bees and Flowers

Often scientists use rate of change equations in their study of population growth for one or more species. In this problem we study systems of rate of change equations designed to inform us about the future populations for two species that are either competitive (that is, both species are *harmed by* interaction) or cooperative (that is, both species *benefit from* interaction).

1. Which system of rate of change equations below describes a situation where the two species compete and which system describes cooperative species? Explain your reasoning.

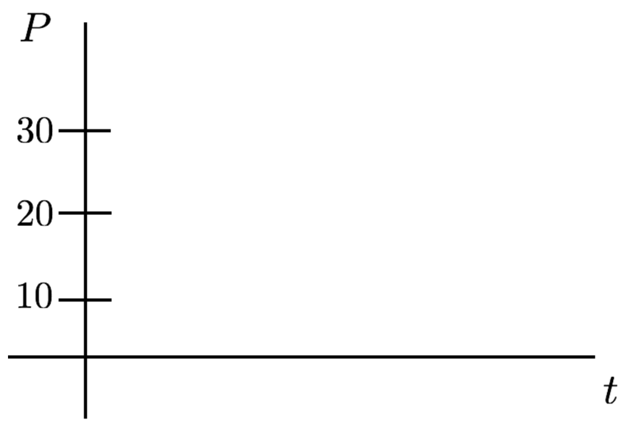
$$\text{(i)}\quad \frac{dx}{dt} = -5x + 2xy \qquad\qquad \text{(ii)}\quad \frac{dx}{dt} = 4x - 2xy$$
$$\frac{dy}{dt} = -4y + 3xy \qquad\qquad \frac{dy}{dt} = 2y - xy$$

A Simplified Situation

The previous problem dealt with a complex situation with two interacting species. To develop the ideas and tools that we will need to further analyze complex situations like these, we will simplify the situation by making the following assumptions:

- There is only one species (*e.g.*, fish)
- The species has been in its habitat (*e.g.*, a lake) for some time prior to what we call $t = 0$
- The species has access to unlimited resources (*e.g.*, food, space, water)
- The species reproduces continuously

2. Given these assumptions for a certain lake containing fish, sketch three possible population versus time graphs: one starting at $P = 10$, one starting at $P = 20$, and the third starting at $P = 30$.



   (a) For your graph starting with $P = 10$, how does the slope vary as time increases? Explain.

   (b) For a set P value, say $P = 30$, how do the slopes vary across the three graphs you drew?

3. This situation can also be modeled with a rate of change equation, $\frac{dP}{dt} = \textit{something}$. What should the "something" be? Should the rate of change be stated in terms of just P, just t, or both P and t? Make a conjecture about the right hand side of the rate of change equation and provide reasons for your conjecture.

What Exactly is a Differential Equation and What are Solutions?

A **differential equation** is an equation that relates an unknown function to its derivative(s). Suppose $y = y(t)$ is some unknown function, then a differential equation, or rate of change equation, would express the rate of change, $\frac{dy}{dt}$, in terms of y and/or t. For example, all of the following are differential equations.

$$\frac{dP}{dt} = kP, \qquad \frac{dy}{dt} = y + 2t, \qquad \frac{dy}{dt} = t^2 + 5, \qquad \frac{dy}{dt} = \frac{6y-2}{ty}, \qquad \frac{dy}{dt} = \frac{y^2-1}{t^2+2t}$$

In particular, these are all examples of *first order* differential equations because only the first derivative appears in the equation. Given a rate of change equation for some unknown function, **solutions** to this rate of change equation are *functions* that satisfies the rate change equation. A constant function that satisfies the differential equation is called an **equilibrium solution**.

4. One way to read the differential equation $\frac{dy}{dt} = y + 2t$ aloud you would say, "dee y dee t equals y plus two times t." However, this does **not** relate to the *meaning* of the solution. How might you read this differential equation *with meaning*?

5. (a) Is the function $y = 1 + t$ a solution to the differential equation $\dfrac{dy}{dt} = \dfrac{y^2-1}{t^2+2t}$? How about the function $y = 1 + 2t$? How about $y = 1$? Explain your reasoning.

   (b) Is the function $y = t^3 + 2t$ a solution to the differential equation $\dfrac{dy}{dt} = 3y^2 + 2$? Why or why not?

6. Figure out all the functions that satisfy the rate of change equation $\dfrac{dP}{dt} = 0.3P$.
   (*Hint*: read the differential equation with meaning.)

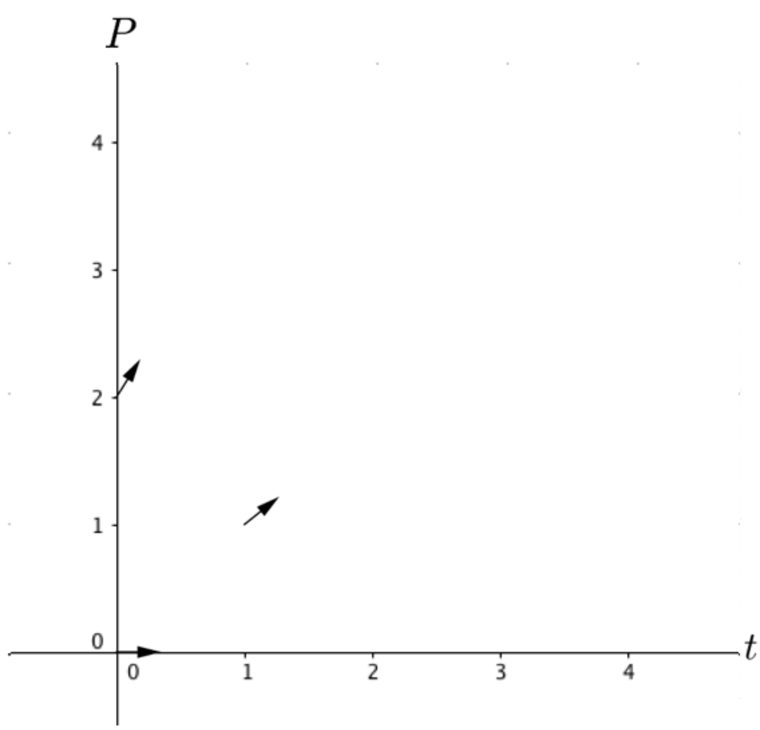
7. Figure out all of the solutions to the differential equation $\dfrac{dy}{dt} = t^2 + 5$.

Slope Fields

A **slope field** is a graphical representation of a rate of change equation. Given a rate of change equation, if we plug in particular values of (t, y) then $\frac{dy}{dt}$ tells you the slope of the tangent vector to the solution at that point.

For example, consider the rate of change equation $\frac{dy}{dt} = y + 2t$. At the point (1, 3), the value of $\frac{dy}{dt}$ is 5. Thus, the slope field for this equation would show a vector at the point (1, 3) with slope 5. A slope field depicts the exact slope of many such vectors, where we take each vector to be uniform length. Slope fields are useful because they provide a graphical approach for obtaining qualitatively correct graphs of the functions that satisfy a differential equation.

8. Below is a partially completed slope field for $\frac{dP}{dt} = 0.8P$.
   (a) Plot many more tangent vectors to create a slope field.
   (b) Use your slope field to sketch in qualitatively correct graphs of the solution functions that start at $P = 0, 0.5$, and 2, respectively. Note: the value of P at an initial time (typically $t = 0$) is called an **initial condition**.
   (c) Recall that a solution to a differential equation is a function that satisfies the differential equation. Explain how the graph with initial condition $P(0) = 1$ can graphically be thought of as a solution to the differential equation when the differential equation is represented by its slope field.

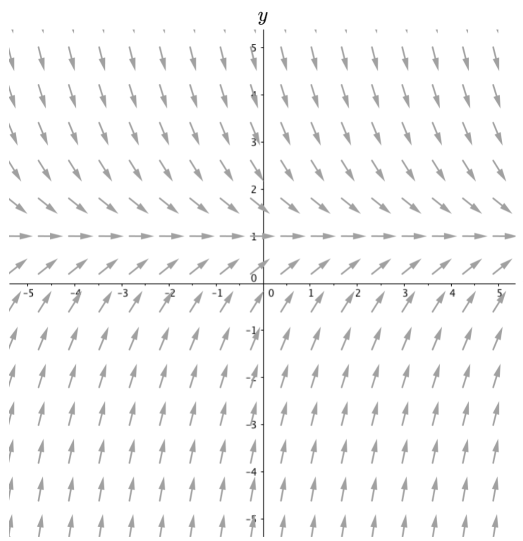


t	P	$\frac{dP}{dt} = 0.8P$
0	0	0
0	2	1.6
1	1	0.8

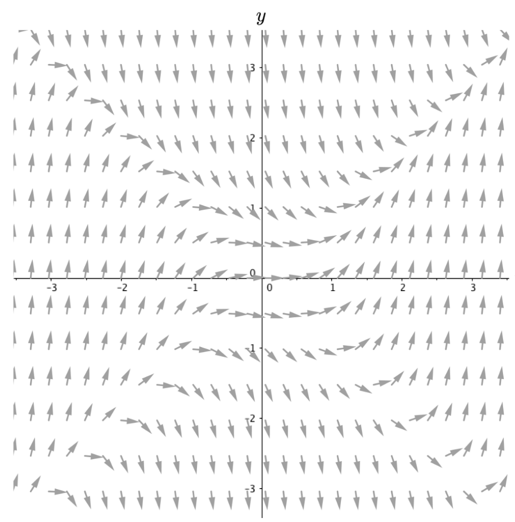
9. Below are seven rate of changes equations and three different slope fields. Without using technology, identify which differential equation is the best match for each slope field (thus you will have four rate of change equations left over). Explain your reasoning.

$$\text{(i) } \frac{dy}{dt} = t - 1 \quad \text{(ii) } \frac{dy}{dt} = 1 - y^2 \quad \text{(iii) } \frac{dy}{dt} = y^2 - t^2 \quad \text{(iv) } \frac{dy}{dt} = 1 - y$$

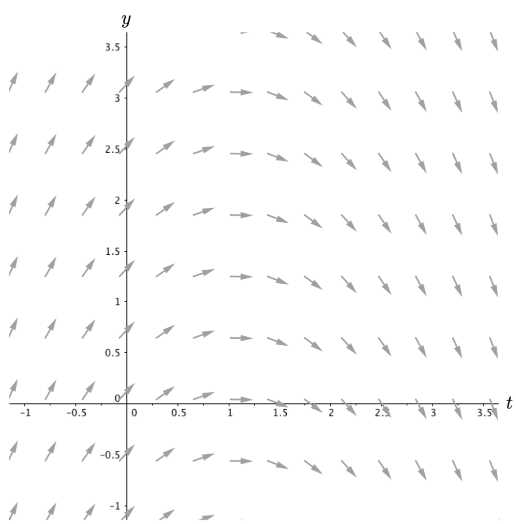
$$\text{(v) } \frac{dy}{dt} = t^2 - y^2 \quad \text{(vi) } \frac{dy}{dt} = 1 - t \quad \text{(vii) } \frac{dy}{dt} = 9t^2 - y^2$$



(a)



(b)



(c)

10. For each of the slope fields in the previous problem, sketch in graphs of several different qualitatively correct solutions.

Homework Set 1

1. Consider the following systems of rate of change equations:

System A

$$\frac{dx}{dt} = 3x\left(1 - \frac{x}{10}\right) - 20xy$$
$$\frac{dy}{dt} = -5y + \frac{xy}{20}$$

System B

$$\frac{dx}{dt} = 0.3x - \frac{xy}{100}$$
$$\frac{dy}{dt} = 15y\left(1 - \frac{y}{17}\right) + 25xy$$

In both of these systems, x and y refer to the number of two different species at time t. In particular, in one of these systems the prey are large animals and the predators are small animals, such as piranhas and humans. Thus it takes many predators to eat one prey, but each prey eaten is a tremendous benefit for the predator population. The other system has very large predators and very small prey.

Figure out which system is which and explain the reasoning behind your decision.

2. Consider the rate of change equation

$$\frac{dy}{dt} = 0.5y(2 + y)(y - 8),$$

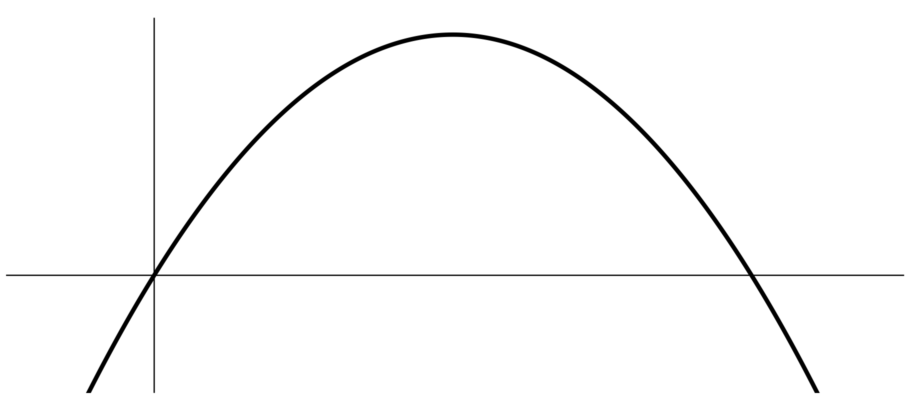
which has been created to provide predictions about the future population of rabbits over time.

(a) For what values of y is $y(t)$ increasing? Explain your reasoning.

(b) For what values of y is $y(t)$ decreasing? Explain your reasoning.

(c) For what values of y is $\frac{dy}{dt}$ neither positive nor negative? What does this imply about the solution function $y(t)$?

3. Valeria created the following graph to help her analyze solutions to the differential equation $\frac{dy}{dt} = 2y\left(1 - \frac{y}{10}\right)$. What is this a graph of (*i.e.*, what are the axes for this graph)? What information about solutions can you glean from this graph?



4. Suppose two students are memorizing the elements on a list according to the rate of change equation

$$\frac{dL}{dt} = 0.5(1 - L),$$

where L represents the fraction of the list that is memorized at any time t.

   (a) If one of the students knows one-third of the list at time $t = 0$ and the other student knows none of the list, which student is learning most rapidly at this instant? Why?

   (b) What does the rate of change equation predict for someone who begins with the list completely memorized? Explain.

   (c) Suppose now that the list so long that no one can read it in a lifetime, like the decimal representation for π. In reality no one can memorize all the digits to π, but what does the rate of change equation predict will happen for a person who starts out not knowing any of the digits? That is, according to the rate of change equation, if $L = 0$ at time $t = 0$, is there ever a value of t for which $L = 1$? Explain.

5. The letter y appears in two places in the differential equation $\frac{dy}{dt} = 0.3y$. Is it appropriate to think of both occurrences of y as function of t? Explain.

6. In algebra, the goal of solving an equation such as $x^2 + 4x = 2$ is to find the values of x that make a true statement. In differential equations, what is the goal of solving an equation such as $\frac{dx}{dt} + 4x = 2$?

7. For the differential equation $\frac{dy}{dt} = 1 - y^2$,

   (a) Sketch a slope field by hand.

   (b) Describe any shortcuts or patterns you used to make the task easier.

   (c) Sketch several $y(t)$ graphs.

8. Differential equations are often referred to as mathematical models. Explain what the phrase “mathematical model” means to you, what previous experiences you have had with mathematical models, and how the mathematical use of the word model is similar to and/or different from the everyday use of the word model (*e.g.*, fashion model, model airplane, model student).

9. Consider the differential equation

$$\frac{dy}{dt} + ty = e^{-\frac{1}{2}t^2}; \quad y(0) = 1$$

   Is $y(t) = te^{-\frac{1}{2}t^2} + e^{-\frac{1}{2}t^2}$ a solution?

10. (a) Go to the glossary and identify all terms that are relevant to this unit and list those terms here.

    (b) Are there other vocabulary terms that you think are relevant for this unit that were not included? If yes, list them.